ON THE WEAK-OPEN IMAGES OF METRIC SPACES

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Abstract. In this paper, we give characterizations of certain weak-open images of metric spaces.

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1. Introduction

To find internal characterizations of certain images of metric spaces is one of the central problems in General Topology. Recently, S. Xia [4] introduced the concept of weak-open mappings. By using it, certain g-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Papers [6], [8], [9], [10], [11], [20] have done some wonderful work on g-metrizable spaces, but have only investigated internal characterizations of g-metrizable spaces. The present paper establishes the relationships between g-metrizable spaces (spaces with compact-countable weak-bases) and metric spaces by means of weak-open mappings, π-mappings and σ-mappings (weak-open mappings and cs-mappings, respectively).

In this paper, all spaces are regular and T_1, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers, ω denotes $\mathbb{N} \cup \{0\}$. For a collection $\mathscr{P}$ of subsets of a space X and a mapping $f\colon X \to Y$, denote $f(\mathscr{P}) = \{f(P)\colon P \in \mathscr{P}\}$.

Definition 1.1. Let $\mathscr{P}$ be a cover of a space X. $\mathscr{P}$ is called compact-countable if for each compact subset K of Y, only countably many members of $\mathscr{P}$ intersect K.

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Definition 1.2. Let $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ be a collection of subsets of a space X satisfying that for each $x \in X$,

(1) $\mathscr{P}_x$ is a network of x in X,

(2) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$.

$\mathscr{P}$ is called a weak-base for X [2] if a subset G of X is open in X if and only if for each $x \in G$, there exists $P \in \mathscr{P}_x$ such that $P \subset G$.

A space X is called a g-metrizable space [3] if X has a σ-locally finite weak-base.

Definition 1.3. Let $f\colon X \to Y$ be a mapping.

(1) f is a weak-open mapping [4] if there exists a weak-base $\mathscr{B} = \bigcup\{\mathscr{B}_y\colon y \in Y\}$ for Y, and for $y \in Y$ there exists $x(y) \in f^{-1}(y)$ satisfying condition $(*)$: for each open neighbourhood U of $x(y)$, $B_y \subset f(U)$ for some $B_y \in \mathscr{B}_y$.

(2) f is a cs-mapping [5] if for each compact subset K of Y, $f^{-1}(K)$ is separable in X.

(3) f is a σ-mapping [1] if there exists a base $\mathscr{B}$ for X such that $f(\mathscr{B})$ is a σ-locally finite collection of subsets of Y.

(4) f is a π-mapping [19] if (X, d) is a metric space, and for each $y \in Y$ and its open neighbourhood V in Y, $d(f^{-1}(y), X \setminus f^{-1}(V)) > 0$.

It is easy to check that a weak-open mapping is a quotient mapping.

2. The weak-open σ-image of a metric space

Lemma 2.1 [6]**.** *Suppose (X, d) is a metric space and $f\colon X \to Y$ is a quotient mapping. Then Y is a symmetric space if and only if f is a π-mapping.*

Theorem 2.2. *The following are equivalent for a space X:*

(1) *Y is a g-metrizable space.*

(2) *Y is a weak-open, π, σ-image of a metric space.*

(3) *Y is a weak-open σ-image of a metric space.*

Proof. (1) $\Rightarrow$ (2) Suppose Y is a g-metrizable space, then Y has a σ-locally finite weak-base. Let $\mathscr{P} = \bigcup\{\mathscr{P}_i\colon i \in \mathbb{N}\}$ be a σ-locally finite weak-base for Y, where each $\mathscr{P}_i = \{P_\alpha\colon \alpha \in A_i\}$ is locally finite in Y which is closed under finite intersections and $Y \in \mathscr{P}_i \subset \mathscr{P}_{i+1}$. For each $i \in \mathbb{N}$, endow A_i with discrete topology. Then A_i is a metric space. Put

$$X = \Big\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i\colon \{P_{\alpha_i}\colon i \in \mathbb{N}\} \subset \mathscr{P} \text{ forms a network at some point } x(\alpha) \in Y\Big\},$$

and endow X with the subspace topology induced from the usual product topology of the collection $\{A_i\colon i \in \mathbb{N}\}$ of metric spaces. Then X is a metric space. Since Y is Hausdroff, $x(\alpha)$ is unique in Y for each $\alpha \in X$. We define $f\colon X \to Y$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in X$. Because $\mathscr{P}$ is a σ-locally finite weak-base for Y, we couclude that f is surjective. For each $\alpha = (\alpha_i) \in X$, $f(\alpha) = x(\alpha)$. Suppose V is an open neighbourhood of $x(\alpha)$ in Y. Then there exists $n \in \mathbb{N}$ such that $x(\alpha) \in P_{\alpha_n} \subset V$. If we set $W = \{c \in X\colon \text{the n-th coordinate of } c \text{ is } \alpha_n\}$, then W is an open neighbourhood of α in X and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a weak-open σ-mapping.

(i) f is a σ-mapping.

For each $n \in \mathbb{N}$ and $\alpha_n \in A_n$, put

$$V(\alpha_1, \ldots, \alpha_n) = \{\beta \in X\colon \text{for each } i \leqslant n, \text{ the } i\text{-th coordinate of } \beta \text{ is } \alpha_i\}.$$

It is easy to check that $\{V(\alpha_1, \ldots, \alpha_n)\colon n \in \mathbb{N}\}$ is a locally neighbourhood base of α in X.

Let $\mathscr{B} = \{V(\alpha_1, \ldots, \alpha_n)\colon \alpha_i \in A_i \ (i \leqslant n) \text{ and } n \in \mathbb{N}\}$; then $\mathscr{B}$ is a base for X. To prove that f is a σ-mapping, we only need to check that $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i \leqslant n} P_{\alpha_i}$ for each $n \in \mathbb{N}$ and $\alpha_n \in A_n$ because $f(\mathscr{B})$ is σ-locally finite in Y by this result. For each $n \in \mathbb{N}$, $\alpha_n \in A_n$ and $i \leqslant n$ we have $f(V(\alpha_1, \ldots, \alpha_n)) \subset P_{\alpha_i}$, hence $f(V(\alpha_1, \ldots, \alpha_n)) \subset \bigcap\limits_{i \leqslant n} P_{\alpha_i}$. On the other hand, for each $x \in \bigcap\limits_{i \leqslant n} P_{\alpha_i}$ there is $\beta = (\beta_j) \in X$ such that $f(\beta) = x$. For each $j \in \mathbb{N}$, $P_{\beta_j} \in \mathscr{P}_j \subset \mathscr{P}_{j+n}$, hence there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$, then $\alpha \in V(\alpha_1, \ldots, \alpha_n)$ and $f(\alpha) = x$. Thus $\bigcap\limits_{i \leqslant n} P_{\alpha_i} \subset f(V(\alpha_1, \ldots, \alpha_n))$, hence $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i \leqslant n} P_{\alpha_i}$. Therefore, f is a σ-mapping.

(ii) f is a weak-open mapping.

Denote $\mathscr{P}_y = \{P \in \mathscr{P}\colon y \in P\}$; then $\mathscr{P} = \bigcup\{\mathscr{P}_y\colon y \in Y\}$.

For each $y \in Y$, by is the idea $\mathscr{P}$, there exists $(\alpha_i) \in \bigcap\limits_{i \in \mathbb{N}} A_i$ such that $\{P_\alpha\colon i \in \mathbb{N}\} \subset \mathscr{P}$ is a network of y in Y, hence $\alpha = (\alpha_i) \in f^{-1}(y)$.

Suppose G is an open neighbourhood of α in X. Then there exists $j \in \mathbb{N}$ such that $V(\alpha_1, \ldots, \alpha_j) \subset G$. Thus $f(V(\alpha_1, \ldots, \alpha_j)) \subset f(G)$. By (i), $f(V(\alpha_1, \ldots, \alpha_j)) = \bigcap\limits_{i \leqslant j} P_{\alpha_i}$. So $P_y \subset \bigcap\limits_{i \leqslant j} P_{\alpha_i}$ for some $P_y \in \mathscr{P}_y$. Hence $P_y \subset f(G)$.

Hence there exists a weak-base $\mathscr{P}$ for Y and $\alpha \in f^{-1}(y)$ satisfying the condition $(*)$ from Definition 1.3(1). Therefore f is a weak-open mapping.

(iii) f is a π-mapping.

By (ii), f is a quotient mapping. Since a g-metrizable space is symmetric, f is a π-mapping by Lemma 2.1.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Suppose Y is the image of a metric space X under a weak-open σ-mapping f. Since f is a σ-mapping, there exists a base $\mathscr{B}$ for X such that $f(\mathscr{B})$ is σ-locally finite in Y. And since f is a weak-open mapping, there exists a weak-base $\mathscr{P} = \bigcup\{\mathscr{P}_y\colon\ y \in Y\}$ for Y such that for each $y \in Y$ there exists $x(y) \in f^{-1}(y)$ satisfying the condition $(*)$ from Definition 1.3(1). For each $y \in Y$, put

$$\mathscr{F}_y = \{f(B)\colon\ x(y) \in B \in \mathscr{B}\},$$
$$\mathscr{F} = \bigcup\{\mathscr{F}_y\colon\ y \in Y\}.$$

Obviously, $\mathscr{F} \subset f(\mathscr{B})$, hence $\mathscr{F}$ is σ-locally finite in Y. We will prove that $\mathscr{F}$ is a weak-base for Y.

It is obvious that $\mathscr{F}$ satisfies the condition (1) from Definition 1.2. For each $y \in Y$, suppose $U, V \in \mathscr{F}_y$; then there exist $B_1 \in \mathscr{B}$ and $B_2 \in \mathscr{B}$ such that $x(y) \in B_1 \cap B_2$ and $f(B_1) = U$, $f(B_2) = V$. Since $\mathscr{B}$ is a base for X, there exists $B \in \mathscr{B}$ such that $x(y) \in B \subset B_1 \cap B_2$. Thus $f(B) \in \mathscr{F}_y$ and $f(B) \subset f(B_1 \cap B_2) \subset U \cap V$. Hence $\mathscr{F}$ satisfies the condition (2) from Definition 1.2.

Suppose $G \subset Y$ is open in Y, then $x(y) \in f^{-1}(G)$ for each $y \in G$. Since $\mathscr{B}$ is a base for X, we have $x(y) \in B \subset f^{-1}(G)$ for some $B \in \mathscr{B}$. Thus $f(B) \in \mathscr{F}_y$ and $f(B) \subset G$. On the other hand, suppose that $G \subset Y$ and for $y \in G$ there exists $F \in \mathscr{F}_y$ such that $F \subset G$. Then there exists $B \in \mathscr{B}$ such that $x(y) \in B$ and $F = f(B)$. Since B is an open neighbourhood of $x(y)$, there exists $P_y \in \mathscr{P}_y$ such that $P_y \subset f(B)$. Thus for each $y \in G$ there exists $P_y \in \mathscr{P}_y$ such that $P_y \subset G$. Hence G is open in Y because $\mathscr{P}$ is a weak-base for Y. So $\mathscr{F}$ is a weak-base for Y.

Therefore Y is a g-metrizable space. □

3. The weak-open *cs*-image of a metric space

Theorem 3.1. *A space Y has a compact-countable weak-base if and only if Y is a weak-open cs-image of a metric space.*

P r o o f. *Sufficiency.* Suppose Y is the image of a metric space X under a weak-open *cs*-mapping f. Since f is a weak-open mapping, there exists a weak-base $\mathscr{B} = \bigcup\{\mathscr{B}_y\colon\ y \in Y\}$ for Y such that for each $y \in Y$ there exists $x(y) \in f^{-1}(y)$ satisfying the condition $(*)$ from Definition 1.3(1). Because X is a metric space, X has a σ-locally finite base. Let $\mathscr{P}$ be a σ-locally finite base for X. For each $P \in \mathscr{P}$, put

$$\mathscr{B}_P = \{B \in \mathscr{B}\colon\ B \subset f(P)\},$$
$$B_P = \bigcup \mathscr{B}_P,$$

then $B_P \subset f(P)$. For each compact subset K of Y, since f is a cs-mapping, $f^{-1}(K)$ is separable in X. So $f^{-1}(K)$ is a Lindelöf subspace of X. Because a locally finite collection of a Lindelöf space is countable, $\{P \in \mathscr{P} \colon P \cap f^{-1}(K) \neq \Phi\}$ is countable. Thus $f(\mathscr{P})$ is compact-countable. Hence $\mathscr{B}^* = \{B_P \colon P \in \mathscr{P}\}$ is compact-countable. For each $y \in Y$, put

$$\mathscr{B}'_y = \{B_P \in \mathscr{B}^* \colon B_y \in \mathscr{B}_P \text{ for some } B_y \in \mathscr{B}_y\},$$
$$\mathscr{B}''_y = \Big\{\bigcap \mathscr{U} \colon \mathscr{U} \text{ is a finite subcollection of } \mathscr{B}'_y\Big\},$$
$$\mathscr{B}'' = \bigcup \{\mathscr{B}''_y \colon y \in Y\},$$

then $\mathscr{B}''$ is compact-countable. We will prove that $\mathscr{B}''$ is a weak-base for Y. It is easy to check that $\mathscr{B}''$ satisfies the condition (1), (2) from Definition 1.2.

Suppose V is open in Y for each $y \in V$, since $\mathscr{P}$ is a base for X, then $x(y) \in P \subset f^{-1}(V)$ for some $P \in \mathscr{P}$. Thus there exists $B_y \in \mathscr{B}_y$ such that $B_y \subset f(P)$, and so $B_y \in \mathscr{B}_P$. Hence $B_P \in \mathscr{B}'_y \subset \mathscr{B}''_y$ and $B_P \subset f(P) \subset V$. On the other hand, suppose $V \subset Y$ is such that for each $y \in V$, $B \subset V$ for some $B \in \mathscr{B}''_y$. By the properties of $\mathscr{B}'$ and $\mathscr{B}''$ and the condition (2) from Definition 1.2, there exists $B_y \in \mathscr{B}_y$ such that $y \in B_y \subset B \subset V$. Because $\mathscr{B} = \bigcup\{\mathscr{B}_y \colon y \in Y\}$ is a weak-base for Y, V is open in Y. Therefore $\mathscr{B}''$ is a weak-base for Y.

Necessity. Suppose $\mathscr{P}$ is a compact-countable weak-base for Y. Endow $\mathscr{P}$ with discrete topology, then $\mathscr{P}$ is a metric space. Put $X = \{(P_n) \in \mathscr{P}^\omega \colon \{P_n \colon n \in \mathbb{N}\}$ is a network of some point $y \in Y\}$, and endow X with the subspace topology induced by the product topology of the usual product space $\mathscr{P}^\omega$. Then X is a metric space. Since Y is Hausdroff, y is unique in Y (in fact, it is easy to check that $\{y\} = \bigcap\limits_{n \in \mathbb{N}} P_n$). We define $f \colon X \to Y$ by $f((P_n)) = y$ for each $(P_n) \in X$. For each $y \in Y$, since $\mathscr{P}$ is point-countable in Y, denoting $\{P \in \mathscr{P} \colon y \in P\}$ by (P_n), we have $(P_n) \in X$ and $f((P_n)) = y$. Thus f is a surjection. It is obvious that f is continuous. We will prove that f is a weak-open cs-mapping.

(i) f is a weak-open mapping.

For each $y \in Y$, denote a collection of weak neighbourhoods of y in Y by $\mathscr{P}_y$; then $\mathscr{P}_y$ is countable. Set $\mathscr{P}_y = \{P_n \colon n \in \mathbb{N}\}$, then $f((P_n)) = y$ and $(P_n) \in f^{-1}(y)$. For each $n \in \mathbb{N}$, put

$$B(P_1, \ldots, P_n) = \{(P'_n) \in X \colon P'_i = P_i \text{ for each } i \leqslant n\}.$$

It is easy to check that $\{B(P_1, \ldots, P_n) \colon n \in \mathbb{N}\}$ is a locally neighbourhood base of the point (P_n) in X. □

Claim. $f(B(P_1,\ldots,P_n))=\bigcap\limits_{i\leqslant n}P_i$ *for each* $n\in\mathbb{N}$.

Suppose $(P_i')\in B(P_1,\ldots,P_n)$, then $f((P_i'))=\bigcap\limits_{i\in\mathbb{N}}P_i'\subset\bigcap\limits_{i\leqslant n}P_i$. Thus $f(B(P_1,\ldots,P_n))\subset\bigcap\limits_{i\leqslant n}P_i$. On the other hand, suppose $z\in\bigcap\limits_{i\leqslant n}P_i$ and set $\mathscr{P}_z=\{P''_{n+j}\colon j\in\mathbb{N}\}$. Put

$$P_r^*=\begin{cases}P_r, & r\leqslant n,\\ P_r'', & r>n,\end{cases}$$

then $(P_r^*)\in B(P_1,\ldots,P_n)$ and $f((P_r^*))=z$. Thus $\bigcap\limits_{i\leqslant n}P_i\subset f(B(P_1,\ldots,P_n))$. Hence $f(B(P_1,\ldots,P_n))=\bigcap\limits_{i\leqslant n}P_i$.

Because $\mathscr{P}$ is a weak-base for Y and $\{P_n\colon n\in\mathbb{N}\}=\mathscr{P}_y$, we obtain $f(B(P_1,\ldots,P_n))=\bigcap\limits_{i\leqslant n}P_i\in\mathscr{P}_y$ for each $n\in\mathbb{N}$. Suppose G is a open neighbourhood of the point (P_n) in X; then there exists $j\in\mathbb{N}$ such that $B(P_1,\ldots,P_j)\subset G$. So $f(B(P_1,\ldots,P_j))\subset f(G)$. By the Claim, $f(B(P_1,\ldots,P_j))=\bigcap\limits_{i\leqslant j}P_i\in\mathscr{P}_y$. Hence there exists a weak-base $\mathscr{P}$ for Y and $(P_n)\in f^{-1}(y)$ satisfying the condition $(*)$ from Definition 1.3(1). Therefore f is a weak-open mapping.

(ii) f is a *cs*-mapping.

For each compact subset K of Y, since $\mathscr{P}$ is compact-countable, hence $\{P\in\mathscr{P}\colon P\cap K\neq\Phi\}$ is countable. Thus $\{P\in\mathscr{P}\colon P\cap K\neq\Phi\}^\omega\cap X$ is a hereditarily separable subspace of X. Because $f^{-1}(K)\subset\{P\in\mathscr{P}\colon P\cap K\neq\Phi\}^\omega\cap X$, thus $f^{-1}(K)$ is separable in X. Hence f is a *cs*-mapping.

Remark 3.2. A mapping $f\colon X\to Y$ is an s-mapping (ss-mapping [16]) if for each $y\in Y$, $f^{-1}(y)$ is separable in X (for each $y\in Y$, there exists an open neighbourhood V of y in Y such that $f^{-1}(V)$ is separable in X). A mapping $f\colon X\to Y$ is a 1-sequence-covering mapping [14] if for each $y\in Y$ there exists $x\in f^{-1}(y)$ satisfying the following condition: whenever $\{y_n\}$ is a sequence in Y converging to a point y in Y, there exists a sequence $\{x_n\}$ of X converging to a point x in X such that each $x_n\in f^{-1}(y_n)$. Obviously, if X is a metric space, then an ss-mapping $\Rightarrow$ a cs-mapping $\Rightarrow$ an s-mapping. However, we have the following facts.

Example 1. A weak-open s-image of a metric space is not a weak-open cs-image of a metric space.

Let

$$S=\left\{\frac{1}{n}\colon n\in\mathbb{N}\right\}\cup\{0\},\quad X=[0,1]\times S,$$

and let

$$Y=[0,1]\times\left\{\frac{1}{n}\colon n\in\mathbb{N}\right\}$$

have the usual Euclidean topology as a subspace of $[0,1]\times S$. Define a typical neighbourhood of $(t,0)$ in X to be of the form

$$\{(t,0)\}\cup\Big(\bigcup_{k\geqslant n}V(t,1/k)\Big),\quad n\in\mathbb{N},$$

where $V(t,1/k)$ is a neighbourhood of $(t,1/k)$ in $[0,1]\times\{1/k\}$. Put

$$M=\Big(\bigoplus_{n\in\mathbb{N}}[0,1]\times\{1/n\}\Big)\oplus\Big(\bigoplus_{t\in[0,1]}\{t\}\times S\Big)$$

and define f from M onto X such that f is an obvious mapping.

Then f is a compact-covering, quotient, two-to-one mapping from the locally compact metric space M onto the separable, regular, non-Lindelöf, k-space X (see Example 2.8.16 of [13] or Example 9.3 of [18]). It is easy to check that f is a 1-sequence-covering mapping. By Theorem 2.5 of [14], X has a point-countable weak-base. Thus X is a weak-open s-image of a metric space by Theorem 2.5 of [4].

X has no compact-countable k-network. Indeed, suppose $\mathscr{P}$ is a compact-countable k-network for X. Put

$$\mathscr{F}=\{\{(t,0)\}\colon\ t\in[0,1]\}\cup\{P\cap Y\colon\ P\in\mathscr{P}\}.$$

Since $[0,1]\times\{0\}$ is a closed discrete subspace of X, $\mathscr{F}$ is a k-network for X. But Y is a σ-compact subspace of X. Thus $\{P\cap Y\colon\ P\in\mathscr{P}\}$ is countable, and so $\mathscr{F}$ is star-countable. Since a regular k-space with a star-countable k-network is an $\aleph_0$-space (see [17]), hence X is a Lindelöf space, a contradiction. Thus X has no compact-countable k-network. By Lemma 7 of [15], X has no compact-countable weak-base. Hence X is not a weak-open cs-image of a metric space by Theorem 3.1.

Example 2. A weak-open cs-image of a metric space is not a weak-open ss-image of a metric space.

Let X be a paracompact space with a point-countable base and not metrizable. Then X has a compact-countable base, and so X has a compact-countable weak-base. By Theorem 3.1, X is a weak-open cs-image of a metric space. But X is not a 1-sequence-covering ss-image of a metric space because X is not a metric space. Thus X is not a weak-open ss-image of a metric space by Proposition 3.3 of [5].

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