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HOMOGENEITY, TERMINALITY AND SOME MAPPING PROBLEMS

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Abstract The collection of terminal subcontinua in the theory of homogeneous continua and some problems concerning atomic mapping, cell-like mapping connected with homogeneity are studied.

A continuum is a compact connected nonempty metric space and a curve is a 1-dimensional continuum. $C(X)$ is the hyperspace of all nonempty subcontinua of a compact space X.

A space X is called homogeneous, if for every pair of points x and y, there exists a homeomorphism $h \in H(X)$ (the autohomeomorphism group is denoted by $H(X)$) such that $h(x) = y$.

A continuum X is decomposable, if it is a union of its two proper subcontinua. Otherwise, X is called indecomposable. A continuum X is called hereditarily decomposable or indecomposable, if every subcontinuum of X is decomposable or indecomposable respectively.

A continuum $Z \subset X$ is said to be terminal provided that if $Y \in C(X)$ such that $Y \cap Z \neq \varnothing$, then $Z \subset Y$ or $Y \subset Z$. Z is called a maximal terminal subcontinuum of X, if except X there is no any terminal subcontinuum properly containing Z. We will denote the collection of all terminal subcontinua of X by $T(X)$ and the collection of all indecomposable subcontinua of X by $IN(X)$.

Following propositions are the basic facts about $T(X)$.

Proposition 1[1]. If X is a homogeneous continuum, then

$$T(X) \setminus \{X\} \subset IN(X).$$

Proposition 2[1]. If a continuum X is homogeneous and $Y \in C(X) \setminus T(X)$, then the maximal

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terminal continua contained in Y form a completely regular monotone decomposition.

A continuous mapping $f: X \to Y$ from continuum X onto Y is said to be atomic mapping[2], if for every subcontinuum K of X such that $f(K)$ is nondegenerate, then

$$K = f^{-1}f(K).$$

Lemma 3. A continuous surjective mapping $f: X \to Y$ from continuum X onto Y is atomic if and only if each point inverse $f^{-1}(y)$ is a terminal continuum of X.

A point $a \in A \in C(X)$ is called an outlet point of A if $a \in Z$ for every $Z \in C(X)$ such that $Z \cap A \neq \varnothing \neq Z - A$[3]. The set of all outlet points of A will be denoted by $F(A)$. It is clear that T is terminal if and only if $T = F(T)$.

We need following propositions.

Proposition 4. The terminality of a subcontinuum $Z \in C(X)$ and the maximality of a terminal subcontinuum Z of X are all intrinsic topological invariants, i.e., they are invariant under homeomorphisms of X onto itself.

Proposition 5[3]. If $f: X \to Y$ is a confluent mapping, then $f(F(A)) \subset F(f(A))$ for every $A \in C(X)$.

Proposition 6[4]. Let f be a continuous mapping from X onto Y. The following conditions are equivalent:

(1) f is atomic,

(2) the inverse image of any terminal subcontinuum of Y is a terminal subcontinuum of X,

(3) f is monotone and, for every subcontinuum K of X such that the set $f(K)$ is nondegenerate, we have $K = f^{-1}(f(K))$.

From Propositions 5 and 6 it is easy to obtain the following proposition.

Proposition 7. Let $f: X \to Y$ be a surjective monotone continuous mapping, then f is atomic if and only if the terminality of any subcontinuum is preserved under f and the inverse imaging.

Proof. Because atomic mapping is confluent, it is enough to see that if $A \in C(X)$ is terminal, then $f(A) = f(F(A)) \subset F(f(A))$ and $F(f(A)) \subset f(A)$.

It is from Proposition 1 that for a homogeneous continuum every terminal subcontinuum is indecomposable. But the inverse is not true in general. For example, denote unit circle by S^1 and pseudo-arc by P. Let $X = S^1 \times P$, which is a homogeneous continuum. Take a point $s_0 \in S^1$, $\{s_0\} \times P$ is indecomposable but it is not a terminal subcontinuum of X since $S^1 \times \{p_0\} \setminus \{s_0\} \times P \neq \varnothing \neq \{s_0\} \times P \setminus S^1 \times \{p_0\}$ and the intersection is nonempty where $p_0 \in P$. This is connected with $\dim X > 1$. Therefore, the indecomposability is an absolute concept but the terminality is a relative intrinsic concept.

For homogeneous hereditarily decomposable continua and homogeneous hereditarily indecomposable continua it is clear that $IN(X) = T(X) \setminus \{X\}$ and $IN(X) = T(X)$ respectively.

There is a question arising in [5]: Does it follow that the atomic image of a homogeneous continuum is homogeneous?

Theorem 8. Suppose that X is a homogeneous continuum and $f: X \to Y$ is a surjective atomic mapping and the collection $\mathbf{A} = \{f^{-1}(y); y \in Y\}$ satisfies a condition: $\forall y \neq y'$ and $h \in H(X)$, $h(f^{-1}(y)) \cap f^{-1}(y') \neq \varnothing$ implies $h(f^{-1}(y)) = f^{-1}(y')$. Then Y is a homogeneous continuum.

In fact, $\mathbf{A} = \{f^{-1}(y); y \in Y\} \subset C(X)$ in Theorem 8 forms a principal anti-chain in 2^X. Particularly, by Lemma 3 and Proposition 4, when every point inverse $f^{-1}(y)$ is a maximal terminal subcontinuum, then previous condition is satisfied.

The proof of this theorem is not difficult and omitted here.

Without this condition there will be a counterexample. Let X be the circle of pseudo-arcs that is a homogeneous decomposable continuum and there is a continuous decomposition of X into pseudo-arcs such that the decomposition space is a simple closed curve. Take a maximal terminal subcontinuum T_0 which is a pseudo-arc, one can define a quotient mapping f on X such that $f(T_0)$ is a single point and f is injective on $X \setminus T_0$. The image is given a quotient topology. Then this mapping is an atomic mapping and it is not difficult to see that the quotient space $f(X)$ is not homogeneous.

From the Terminal Decomposition Theorem[6], Proposition 6.1, Theorem 6.6 of [3] and Proposition 1 we also have

Proposition 9. Let X be a homogeneous curve and a point $a \in A \in C(X)$. Then A is a terminal subcontinuum if and only if there exists a unique order arc $[a, A]$ in 2^X. Moreover, if $f: X \to Y$ is an atomic mapping and a point inverse $f^{-1}(y) \subset A$, then $[f^{-1}(y), A]$ is order-isomorphic to $[y, f(A)]$.

A continuum X is cell-like if each mapping of X into a compact ANR is inessential. If the mapping is inessential, we write $f \simeq 0$.

A mapping is cell-like if each of its point inverses is cell-like.

Proposition 10. (Theorem 1 of [7]). If A is a terminal subcontinuum of the continuum X, if B is a subcontinuum of X disjoint from A, and if $f: A \to Y$ is a map of A into the ANR Y, then there exists a map $F: X \to Y$ such that $F|_A = f$ and $F|_B \simeq 0$.

Theorem 11. Suppose that X is a homogeneous continuum and A is a terminal subcontinuum of X, then A is cell-like.

Proof. If X is hereditarily indecomposable, then there exists a terminal decomposition of X (see [7]) such that A is an element of the decomposition. Therefore, by the Theorem 4 of [7] A is cell-like. If X is not hereditarily indecomposable, there exists a maximal terminal subcontinuum K containing A. The subcontinuum K is an element of a maximal terminal decomposition of X (see [6]). From Theorem 4 of [7] we know that K is cell-like. Let $f: A \to Z$ be any continuous mapping into a compact ANR Z and suppose $F: K \to Z$ is an extension of f. Since K is cell-like, hence $F \simeq 0$. By Theorem 12.35, of "Continuum Theory"

(p. 256) by Sam B. Nadler, Jr, we get that $f \simeq 0$. It follows that A is also a cell-like set.

Corollary 12 Suppose that $f: X \to Y$ is an atomic mapping of a homogeneous continuum X onto a nondegenerate continuum Y, then Y is a cell-like mapping.

In [1] Mackowiak defined that a terminal subcontinuum Q of X is a beginning of a jump if $Q \subseteq L \subseteq K \neq Q, Q, L, K \in T(X)$ imply $Q = L$ or $L = K$. He showed that if $Q \in T(X)$ is a beginning of a jump in a homogeneous continuum X, then Q is homogeneous.

Corollary 13 Suppose that X is a homogeneous continuum and a terminal subcontinuum A of X is a beginning of a jump, then A is homogeneous tree-like continuum. Furthermore, any subcontinuum of A is a terminal subcontinuum of X.

It follows that there exists at most one jump on any order arc of $C(X)$ from a sigleton to the homogegeneou continuum X.

Proof. It is sufficient to note that a homogeneous cell-like space is tree-like and homogeneous tree-like space is hereditarily indecomposable.

Let O be an open symmetric neighborhood basis of identity e of the homeomorphism group $H(X)$, i.e., if $o \in O$, then o is an open subset of $H(X)$ such that $o = o^{-1}$ (i.e., $h \in o$ iff $h^{-1} \in o$) and $e \in o$. For each $o \in O$, let H_0 be the subgroup of $H(X)$ generated by o. It is known that H_0 is a closed-open subgroup of $H(X)$. Denote $H = \bigcap_{o \in O} H_0$ each element of which can be represented by a product of finite number of arbitrarily small homeomorphisms.

Theorem 14 Suppose that X is a compact metric space and A is a terminal subcontinuum of X. If there exists a homeomorphism $h \in H$ such that $A \cap h(A) = \varnothing$, then A is cell-like.

Proof. Let Z be a compact ANR. It is well known that there exists an $\epsilon' > 0$ such that for any space W and any two continuous maps $\alpha, \beta: W \to Z$, $\hat{d}(\alpha, \beta) < \epsilon'$ implies $\alpha \simeq \beta$ where $\hat{d}$ is the sup metric on Z^W.

Denote $h(A) = B$, then A is a terminal subcontinuum of X and B is a continuum disjoint from A. Suppose that $g: A \to Z$ is a continuous mapping, by Proposition 10 g has an extension $G: X \to Z$ such that $G|_A = g$ and $G|_B \simeq 0$. For $\epsilon' > 0$, take $\epsilon > 0$ such that for any pair x, x' of X, if $\rho(x, x') < \epsilon$, then $d(G(x), G(x')) < \epsilon'$.

Since $h \in H$, there exist homeomorphisms $h_1, \ldots, h_n$ such that $h^{-1} = h_n \circ \ldots \circ h_1$ and $\hat{\rho}(h_i, e) < \epsilon$ for $1 \leq i \leq n$ where e is an identity. For $x \in B$, since $\rho(x, h_1(x)) < \epsilon$, hence

$$d(G(x), G(h_1(x))) < \epsilon'.$$

Thus

$$\hat{d}(G|_B, G|_{h_1(B)} \circ h_1) < \epsilon'.$$

Similarly,

$$\hat{d}(G|_{h_1(B)}, G|_{h_2 \circ h_1(B)} \circ h_2) < \epsilon',$$

$$\vdots$$

$$\hat{d}(G|_{h_{n-2} \circ \ldots \circ h_1(B)}, G|_{h_{n-1} \circ \ldots \circ h_1(B)} \circ h_{n-1}) < \epsilon',$$

$$\hat{d}(G|_{h_{n-1} \circ \ldots \circ h_1(B)}, G|_{h_n \circ \ldots \circ h_1(B)} \circ h_n) < \epsilon'.$$

By the property of the compact ANR Z,

$$G|_B \simeq G|_{h_1(B)} \circ h_1,$$

$$G|_{h_1}(B) \simeq G|_{h_2 \circ h_1(B)} \circ h_2,$$

$$\dots$$

$$G|_{h_{n-1} \circ \dots \circ h_1(B)} \simeq G|_{h_n \circ \dots \circ h_1(B)} \circ h_n.$$

Because $G|_B \simeq 0$ and $h_1, \dots, h_n$ are homeomorphisms, it follows that $G|_{h_1(B)} \simeq 0$, $G|_{h_2 \circ h_1(B)} \simeq 0, \dots, g = G|_A = G|_{h_n \circ \dots \circ h_1(B)} \simeq 0$. Therefore A is cell-like.

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