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Isolated Chain Recurrent Points of Some Continua

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Abstract For a continuous map of tree or graph to itself, we show the properties that every isolated chain recurrent point is an eventually periodic point, and an isolated chain recurrent point which is not in the orbit of a critical point and has no critical point in its orbit is a periodic point. Furthermore, the property can also be extended to some special λ-dendroids.

Key words continuum; tree; chain recurrent point; eventually periodic point; λ-dendroid

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Chain recurrent points, as one of important concepts in discrete dynamical system, was first introduced by L. Block and J. E. Franke in [1]. In computer iterations it is quite easy to mistake chain recurrent points as periodic points, so it is necessary to make clear the relationships between them. In [2], L. Block and J. E. Franke proved that for a continuous map of the interval to itself or of the circle to itself, any isolated chain recurrent point is an eventually periodic point, and an isolated chain recurrent point which is not in the orbit of a critical point and has no critical point in its orbit is a periodic point. Some results of dynamical system on intervals are still true on some particular continua, for example trees[5,6]. The aim of this paper is to try to generalize the above result from interval maps or circle maps to maps of some continua.

In this paper all maps are continuous. Let (X,d) be a compact metric space, and $f: X \to X$ be a map, f^n is the n-th iteration of f for $n = 0,1,2,\cdots$, and $\mathrm{Per}(f)$ is the set of periodic points of f. Assume $x, y \in X$. For every $\varepsilon > 0$, an ε-chain from x to y is a finite sequence $\{x_0 = x, x_1, \cdots, x_n = y\}$ of X with $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \le i \le n$, and denote that $CR_\varepsilon(x) = \{y \in X : \text{there is an } \varepsilon\text{-chain from } x \text{ to } y\}$. x can be chained to y if for every $\varepsilon > 0$ there is an ε-chain from x to y, and x is called a chain recurrent point if x can be chained to x, and denote that $CR(f) = \{x \in X : x \text{ is a chain recurrent point of } X\}$. The set $\mathrm{Orb}_f(x) = \{f^n(x) : n = 0,1,2,\cdots\}$ is called the orbit of x, $\omega(x)$ denotes the set of limit points of $\mathrm{Orb}_f(x)$. A point x in X is an eventually periodic point if some element in $\mathrm{Orb}_f(x)$ is a periodic point, and x is a critical point of f if f is not a local homeomorphism at x.

By a continuum we mean a compact connected metric space[4]. A dendrite is a locally connected continuum containing no simple closed curve. A graph is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only

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in one or both of their end points. A tree means an acyclic graph, i.e., a graph that contains no simple closed curve. Every tree is a dendrite. Let T denote a tree. We shall first consider the tree map $f: T \to T$. For each $a, b \in T, [a, b]$ denotes the smallest subtree containing $\{a, b\}$ in T.

Lemma 1 For any $x \in T$ and any $\varepsilon > 0$, $CR_\varepsilon(x)$ has a finite number of components, and each component is an open connected subset of T which contains at least one element of the orbit of x.

Proof For each $x \in T$, define $U_\varepsilon(x) = \{y \in T : \text{diam}[x, y] < \varepsilon\}$. Let C be a connected subset of T, since $\text{cl}(C)$ is a subtree of T, which has at most finitely many endpoints, say $a_1, a_2, \cdots, a_n$, put $U_\varepsilon(C) = C \bigcup (\bigcup_{i=1}^n U_\varepsilon(a_i))$. Obviously, $U_\varepsilon(C)$ is an open connected subset of T. Define that $B_1 = U_\varepsilon(f(x))$, and $B_{n+1} = U_\varepsilon(B_n)$ for each $n \geq 1$, then each B_n is an open and connected subset of $T, f^n(x) \in B_n$ and $CR_\varepsilon(x) = \bigcup_{n \geq 1} B_n$, thus each component of $CR_\varepsilon(x)$ is the union of some $B_n' s$, which is open in T and contains at least one element of the orbit of x. If $CR_\varepsilon(x)$ has infinitely many components, denote that $CR_\varepsilon(x) = \bigcup_{j \geq 1} A_j$, here each A_j is a component of $CR_\varepsilon(x)$, then each A_j is open in T and contains at least one element of the orbit of x. For each $j \in N$ there is $n_j \in N$ such that $f^{n_j}(x) \in A_j$. Since T is compact, assume that z is a limit point of the set $\{f^{n_j}(x) : j \in N\}$, then $z \in CR_\varepsilon(x)$, so $z \in A_i$ for some $i \in N$. As A_i is open in T, $f^{n_k}(x) \in A_i$ for some $k > i$, then $A_i \cap A_k \neq \emptyset$, a contradiction. This completes the proof of Lemma 1.

The following two Lemmas hold for every compact metric space X.

Lemma 2[1,2] Let $f: X \to X$. If $x \in CR(f)$, then $f^k(x)$ can be chained to x for each $k \in N$.

Lemma 3[1,2] Let $f: X \to X$. If $x \in CR(f)$ and $z \in \omega(x)$, then z can be chained to x.

Lemma 4 Let $f: T \to T$, and $x \in CR(f)$. Suppose that x is an endpoint of an open connected subset $J \subset T \backslash CR(f)$ such that x can be chained to each point of J, then x is an eventually periodic point.

Proof Suppose that x is not an eventually periodic point, then $\{f^n(x) : n \in N\}$ is an infinite set, and there is a subsequence $\{f^{n_i}(x)\}$ of $\{f^n(x)\}$ with $f^{n_i}(x) \to z \in \omega(x)$. It follows from Lemma 3 that z can be chained to each point of J. Note that $f^n(J) \cap \text{Orb}_f(x) = \emptyset$ for $n = 0, 1, 2, \cdots$, otherwise, by Lemma 2 there is a point in J to be a chain recurrent point. As $x \in \text{cl}(J)$, $f^i(x) \in f^i(\text{cl}(J)) \backslash f^i(J) \subset Bd(f^i(\text{cl}(J)))$. Since T is a tree, we can assume that all $f^{n_i}(x) \in I$ for some closed interval $I \subset T$. While each $\text{cl}(f^i(J))$ is a subtree of T and $f^i(x)$ is one of endpoint of $\text{cl}(f^i(J))$, so $f^{n_i}(J)$ must converge to z. Thus for each point $y \in J$, $z \in \omega(y)$ and so $y \in CR(f)$ contrary to the fact that $J \subset T \backslash CR(f)$.

Lemma 5 Let $f: T \to T$, and $x \in CR(f)$. Then either $x \in \omega(x)$ or x is an endpoint of an open connected subset J of T such that x can be chained to each point of J.

Proof Suppose that $x \notin \omega(x)$, choose $z \in \omega(x)$. Note that $x \notin \omega(z)$, otherwise, $x \in \omega(x)$. Similarly, $x \notin \text{Orb}_f(z)$. So $\delta = d(\{x\}, \text{Orb}_f(z)) > 0$.

For each $n \in N$, by Lemma 1, $CR_{1/n}(z)$ has only finitely many components, and each component contains at least one point of $\text{Orb}_f(z)$. It follows from Lemma 3, there is some component J_n such that $x \in J_n$. Note that the diameter of J_n is at least δ and $J_{n+1} \subset J_n$. Let

$K = \bigcap_{n=1}^{\infty} J_n$. Then z and hence x can be chained to every point of K, and K contains an open connected subset J with one endpoint x.

A space X is said to have the property $(*)$ if for $f : X \to X$, any isolated chain recurrent point of f is an eventually periodic point.

Theorem 1 Every tree has the property $(*)$.

Proof Suppose that x is an isolated chain recurrent point of f. We may assume that x is not a periodic point, then $x \notin \omega(x)$. So by Lemma 5, x satisfies the hypothesis of Lemma 4. Hence x is an eventually periodic point.

Lemma 6[2] Let X be a compact metric space and $f : X \to X$. Suppose $f(p) = p$ and p can be chained to x with $x \neq p$. There is a sequence $\{z_i\}$ approaching p such that for each $i, z_i \neq p$ and z_i can be chained to x.

Theorem 2 Let $f : T \to T$, and let x be an isolated chain recurrent point of f. Suppose that x is not in the orbit of a critical point and no critical point is in the orbit of x, then x is a periodic point.

It can be similarly proved as in [2], we omitted the proof.

Theorem 1 and Theorem 2 also hold for graphs. Though we don't know the above results can be generalized to what kind of dendrites. In the following, we consider some kind of λ-dendroids[4]. Recall some basic concepts on continuum theory. Let X be a continuum. X is said to be hereditarily unicoherent provided that the intersection of any two subcontinua of X is connected. X is said to be hereditarily decomposable provided that every subcontinuum of X is the union of its two proper subcontinua. A λ-dendroid is a hereditarily unicoherent and hereditarily decomposable continuum. Every dendrite is a λ-dendroid. A subcontinuum Q of a continuum X is said to be terminal provided that for every subcontinuum K of X, if $K \cap Q \neq \emptyset$, then either $K \subset Q$ or $Q \subset K$.

Let us introduce a special compactification of trees due to Charatonik[3]. For each tree T and $n \in N$, take a subset $\{q_1, q_2, \cdots, q_n\}$ of T and a finite sequence $\{Q_1, Q_2, \cdots, Q_n\}$ of λ-dendroids with the same finite depth. By Charatonik's construction in [3] there is a compactification $\gamma : T \setminus \{q_1, q_2, \cdots, q_n\} \to \gamma(T \setminus \{q_1, q_2, \cdots, q_n\})$ such that

(1) $X = cl_\gamma(T \setminus \{q_1, q_2, \cdots, q_n\})$ is a λ-dendroid;

(2) The remainder $X \setminus (T \setminus \gamma\{q_1, q_2, \cdots, q_n\})$ consists of n components $Q_1, Q_2, \cdots, Q_n$.

(3) For each $i \in \{1, 2, \cdots, n\}$, the continuum Q_i is a terminal subcontinuum of X.

And if $f : X \to X$ is a surjection, then $f(\bigcup_{i=1}^n Q_i) = \bigcup_{i=1}^n Q_i$. Furthermore, for each Q_i there is exactly one Q_j such that $f(Q_i) = Q_j$. Let $H = X \setminus \bigcup_{i=1}^n Q_i$, then $f(H) = H$. Let $p : X \to T$ be the natural projection, that is, a map such that $p(Q_i) = \{q_i\}$ for $i \in \{1, 2, \cdots, n\}$ and $p_{|H} = \gamma^{-1} : H \to T \setminus \{q_1, q_2, \cdots, q_n\}$ is a one-to-one map, then $p_{|H}$ is a homeomorphism. Define $g : T \to T$ as follows. For each $i \in \{1, 2, \cdots, n\}$, let $g(q_i) = p(f(p^{-1}(q_i)))$, and $g(t) = p(f(p^{-1}(t)))$ for each $t \in T \setminus \{q_1, q_2, \cdots, q_n\}$, then g is well defined and $p \circ f = g \circ p$. The detail properties about the compactification X are discussed in [3]. In the following X always denotes the λ-dendriod defined as above.

Lemma 7 Let $f : X \to X$ be a surjection. Then $x \in CR(f)$ is equivalent to $p(x) \in CR(g)$

for each $x \in H$.

Proof Let d and ρ denote the metric on X and T respectively. Suppose $x \in CR(f)$. Since $p : X \to T$ is uniformly continuous, for any $\varepsilon > 0$, there is some $\delta > 0$ such that $\rho(p(x), p(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Let $\{x_0 = x, x_1, x_2, \cdots, x_n = x\}$ be a δ-chain from x to x in X, set $y_i = p(x_i)$, $0 \le i \le n$, then $\rho(g(y_{i-1}), y_i) = \rho(g(p(x_{i-1})), p(x_i)) = \rho(p(f(x_{i-1})), p(x_i)) < \varepsilon$, so $\{y_0 = p(x), y_1, y_2, \cdots, y_n = p(x)\}$ is an ε-chain from $p(x)$ to $p(x)$ in T, thus $p(x) \in CR(g)$.

Suppose $p(x) \in CR(g)$. Since $p_{|H} : H \to T\backslash\{q_1, q_2, \cdots, q_n\}$ is a homeomorphism, for any $\varepsilon > 0$, there is some $\delta > 0$ such that if y, $z \in T\backslash\{q_1, q_2, \cdots, q_n\}$ and $\rho(y, z) < \delta$, then $d(p^{-1}(y), p^{-1}(z)) < \varepsilon$. For the above δ, there is a $\frac{\delta}{2}$-chain $\{y_0 = p(x), y_1, y_2, \cdots, y_n = p(x)\}$ from $p(x)$ to $p(x)$ in T. If some $y_i \in \{q_1, q_2, \cdots, q_n\}$, since g is uniformly continuous, there exists $\delta' > 0$ such that $\delta' < \frac{\delta}{2}$ and if $\rho(y, z) < \delta'$, then $\rho(g(y), g(z)) < \frac{\delta}{2}$. Choose $y_i' \in T\backslash\{q_1, q_2, \cdots, q_n\}$ such that $\rho(y_i, y_i') < \delta'$, then

$$\rho(g(y_{i-1}), y_i') \le \rho(g(y_{i-1}), y_i) + \rho(y_i, y_i') < \delta,$$

and

$$\rho(g(y_i'), y_{i+1}) \le \rho(g(y_i'), g(y_i)) + \rho(g(y_i), y_{i+1}) < \delta.$$

Replace y_i in the above $\frac{\delta}{2}$-chain by y_i', we can at least get a δ-chain from $y = p(x)$ to $p(x)$ in $T\backslash\{q_1, q_2, \cdots, q_n\}$. Let $x_i = p^{-1}(y_i)$, then clearly $\{x_0 = x, x_1, x_2, \cdots, x_n = x\}$ is an ε-chain from x to x in X, thus $x \in CR(f)$.

Theorem 3 X has the property $(*)$ provided all continua Q_i have the property $(*)$.

Proof Let $f : X \to X$ and set $M(X, f) = \bigcap\{f^m(X) : m \in N\}$. If $M(X, f)$ is contained in some Q_i, then the conclusion follows from the assumption. Otherwise, $M(X, f)$ satisfies all the assumption on X, so we may assume that f is a surjection.

Let x be an isolated chain recurrent point of f, i.e., there is a neighborhood U of x such that $U \cap CR(f) = \{x\}$. Consider the following two cases.

Case 1 $x \in H$. Since H is open in X, we may assume that $U \subset H$. It follows from Lemma 7 that $p(x) \in CR(g)$. $p(x)$ is also an isolated chain recurrent point of g. In fact, since p is a closed map, for the neighborhood U of $x = p^{-1}(p(x))$, there exists a neighborhood V of x in X such that $V \subset U$, $p(V)$ is a neighborhood of $p(x)$ in T and $V = p^{-1}(p(V))$. If $p(V)$ contains a chain recurrent point y other than $p(x)$, then $p^{-1}(y)$ is also a chain recurrent point of f in V other than x, a contradiction. It follows from Theorem 1 that $p(x)$ is an eventually periodic point, i.e., there exists some $k \in N$ such that $g^k(p(x)) \in \mathrm{Per}(g)$. Since $g^k(p(x)) \in T\backslash\{q_1, q_2, \cdots, q_n\}$, and $g^k(p(x)) = p(f^k(x))$, by [3, Proposition 5.3], $f^k(x) \in \mathrm{Per}\,(f)$, so x is an eventually periodic point.

Case 2 $x \in Q_i$ for some $i \in \{1, 2, \cdots, n\}$. There is some $m \in N$ such that $f^m(Q_i) = Q_i$. Since $CR(f) = CR(f^k)$ for each $k \in N$[1], $x \in CR(f) = CR(f^m)$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and if $d(x, y) < \delta$, then $d(f^m(x), f^m(y)) < \frac{\varepsilon}{2}$. For the above δ, there is a δ-chain $\{x_0 = x, x_1, x_2, \cdots, x_n = x\}$ in X such that $d(f^m(x_{i-1}), x_i) < \delta$. If some $x_i \in H$, we can choose $x_i' \in Q_i$ such that $d(x_i, x_i) < \delta$. Then $d(f^m(x_{i-1}), x_i') \le d(f^m(x_{i-1}), x_i) + d(x_i, x_i') <$

$\delta + \delta < \varepsilon$, and $d(f^m(x_i'), x_{i+1}) \le d(f^m(x_i'), f^m(x_i)) + d(f^m(x_i), x_{i+1}) < \frac{\varepsilon}{2} + \delta < \varepsilon$.

So we can get at least an ε-chain from x to x in Q_i. Since Q_i has the property $(*)$, there is $k \in N$ such that $(f^m)^k(x) \in \mathrm{Per}(f^m|_{Q_i}) \subset \mathrm{Per}(f^m) = \mathrm{Per}(f)$, thus x is an eventually periodic point.

Corollary $Sin\frac{1}{x}$-curve has the property $(*)$.

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一些连续统的孤立链回归点

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摘要 对树或图上的连续自映射，本文证明了孤立链回归点是最终周期点，如孤立链回归点不在临界点的轨道中且它的轨道中也不含有临界点，则它还是周期点. 我们还证明了上述性质可以扩张到一类特殊的 λ-dendroid 上.

关键词 连续统；树；链回归点；最终周期点； λ-dendroid