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SPACES WITH COMPACT-COUNTABLE k-SYSTEMS*

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Abstract. In this paper the relations among k-covers, cs^*-covers and k-systems are discussed. The following question is partially answered: Does every separable k'-space with a point-countable k-system have a countable k-system?

1. Introduction

In 1972, E. Michael established the characterizations of paracompact locally compact spaces under quintuple quotient mappings (i.e., open mapping, bi-quotient mapping, countably bi-quotient mapping, pseudo-open mapping and quotient mapping) (see [1]). In 1982, Y. Tanaka investigated spaces with a point-countable k-system [2]. In 1992, S. Lin established the relationships between paracompact locally compact spaces and all kinds of spaces with k-systems [3]. In this paper, we discuss the relations among k-covers, cs^*-covers and k-systems, and partially answer a problem posed by Y. Tanaka [2]. As applications, we give some characterizations for spaces with a compact-countable k-system by means of certain maps on paracompact locally compact spaces, and obtain some corresponding results on locally compact metric spaces.

Let X be a space, and let $\mathcal{P}$ be a cover of X. Then $\mathcal{P}$ is called a k-cover of X if every compact $K \subset X$ is covered by some finite $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is a cs^*-cover of X if for each sequence $\{x_n\}$ converging to $x \in X$, some $P \in \mathcal{P}$ contains the point x and points x_n frequently. Recall some basic definitions. A space X is determined by $\mathcal{P}$ if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is open (closed) in P for every $P \in \mathcal{P}$. If each element of $\mathcal{P}$ is compact (resp. compact metric) in X, then such a cover is called a k-system (resp. mk-system) according to A. V. Arhangel'skii (see [4]). A space X is a k-space (resp. a sequential space), if it is determined by the cover consisting of all (resp. all compact metric) subsets of X. A space X is a k'-space (resp. Fréchet space) if, whenever $x \in \overline{A}$, there exists a compact subset C of X (resp. a sequence $\{a_n : n \in N\}$ in A) with $x \in \overline{A \cap C}$ (resp. $a_n \to x$).

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A collection $\mathcal{P}$ in X is compact-countable (resp. point-countable) if each compact subset of X (resp. each single point) meets only countable many members of $\mathcal{P}$.

A map $f : X \to Y$ is called compact-covering (see [5]) (resp. sequence-covering [6]) if each compact subset (resp. convergent sequence including its limit point) of Y is an image of a compact subset of X under f. A map f is a sequentially quotient map [7] (resp. subsequence-covering map [8]) if for each convergent sequence S of Y, there is a convergent sequence L (resp. compact subset L) of X such that $f(L)$ is a subsequence of S. A map f is called a cL-mapping (resp. cs-mapping [9]) if for any compact subset C of Y, $f^{-1}(C)$ is a Lindelöf (resp. separable) subspace of X. A map f is quotient if whenever $f^{-1}(U)$ is open in X, then U is open in Y. A map f is pseudo-open if whenever $f^{-1}(y) \subset V$ with V open in X, then $y \in \operatorname{int}\big(f(V)\big)$.

In this paper, all spaces are regular and T_1, and all mappings are continuous and onto.

2. Results

PROPOSITION 2.1. *Suppose that $\mathcal{P}$ is a point-countable cover of X. Then $\mathcal{P}$ is a k-system if and only if X is a k-space and $\mathcal{P}$ is a k-cover consisting of compact subsets.*

PROOF. *Necessity*. Since X has a k-system, X is a k-space. So we must prove that $\mathcal{P}$ is a k-cover of X. Suppose not. For each $y \in K$, where K is compact in X, let $(\mathcal{P})_y = \big\{P_i(y) : i \in N\big\}$. Inductively choose $y_n \in K$ such that $y_n \notin P_i(y_j)$ for $i, j < n$. Since K is compact in X, then $A = \{y_n : n \in N\}$ has a cluster point x. Let $L = A \setminus \{x\}$. Then L is not closed in X, and so there is $P \in \mathcal{P}$ such that $P \cap L$ is not closed in X, and hence P contains infinitely many y_n's. Let $P = P_i(y_j)$ for some i and j, then there exists $n > i, j$ such that $y_n \in P_i(y_j)$, a contradiction to the way that the y_n's were chosen.

Sufficiency. Suppose that there exists $F \subset X$ such that $F \cap P$ is closed in X for each $P \in \mathcal{P}$, but F is not closed in X. By the sufficient conditions, $F \cap C$ is not closed in X for some compact $C \subset X$, and so $C \subset \cup\mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$. However, $F \cap C = \cup\big\{(F \cap P) \cap C : P \in \mathcal{P}'\big\}$ is closed in X, a contradiction. Hence X is determined by $\mathcal{P}$, and $\mathcal{P}$ is a k-system for X.

From the proof of Proposition 1.2 in [11], we have:

PROPOSITION 2.2. *Let $\mathcal{P}$ be a point-countable cs^*-cover of X, and let each compact subset of X be a sequential space. Then $\mathcal{P}$ is a k-cover of X.*

PROPOSITION 2.3. *Suppose X is a Fréchet space and $\mathcal{P}$ is a k-cover of X, and $A \subset X$. If $x \in A$, then $x \in \overline{P \cap A}$ for some $P \in \mathcal{P}$.*

PROOF. If $x \in A$, the conclusion is clear. So suppose $x \notin A$. There exists $x_n \in A$ with $x_n \to x$ in X because X is a Fréchet space. Let $K = \{x\} \cup \{x_n : n \in N\}$. Then $K \subset \cup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$, and some $P \in \mathcal{P}'$ must contain infinitely many x_n's, and this P has the required property.

THEOREM 2.4. *Suppose that X is a separable Fréchet space, and a space in which every point is a G_δ. If X has a point-countable k-cover consisting of compact subsets of X, then X has a countable k-cover consisting of compact subsets.*

PROOF. Let Q be a countable dense subset of X, and let $\mathcal{P}$ be a point-countable k-cover consisting of compact subsets of X. Let $\mathcal{R} = \{R : R \in \mathcal{P}$ and $R \cap Q \neq \emptyset\}$. By Proposition 2.3, $\mathcal{R}$ is a countable cover consisting of compact subsets of X. We will show that $\mathcal{R}$ is a k-cover of X. Let K be a compact subset of X and $x \in K$. Put $\mathcal{R} = \{R_n : n \in \omega\}$, where $x \in R_0$. We claim that there exists $n \in \omega$ such that $x \in \operatorname{int}_K \left(\bigcup_{i=0}^{n} R_i \right)$. Suppose not. Since X is a space in which every point is a G_δ, K is a first-countable subspace. So we can choose $x_n \in K \setminus \bigcup_{i \leqq n} R_i$ such that $x_n \to x$. Because each R_n is compact and closed in X, we can also choose $q_{n,k} \in Q \setminus \bigcup_{i \leqq n} R_i$ such that $q_{n,k} \to x_n$ as $k \to \infty$. But then x is in the closure of these $q_{n,k}$'s, so there exists a sequence $q_{n_j,k_j} \to x$ as $j \to \infty$. Since $x_n \neq x$ and $q_{n,k} \neq x$ for all n and k (because $x \in R_0$), we have $n_j \to \infty$ as $j \to \infty$. By Proposition 2.3, some $P \in \mathcal{P}$ contains infinitely many q_{n_j,k_j}'s. Then $P \in \mathcal{R}$, so $P = R_m$ for some $m \in \omega$. But $q_{n_j,k_j} \notin R_m$ when $n_j \geqq m$, a contradiction. Thus $\mathcal{R}$ is a k-cover of X.

COROLLARY 2.5. *Suppose that X is a separable Fréchet space in which every point is a G_δ. If X has a point-countable k-system, then X has a countable k-system.*

PROOF. Let $\mathcal{P}$ be a point-countable k-system for X. By Proposition 2.1, $\mathcal{P}$ is a k-cover consisting of compact subsets of X. In view of Theorem 2.4, X has a countable k-cover $\mathcal{P}'$ consisting of compact subsets. By Proposition 2.1, $\mathcal{P}'$ is a countable k-system.

REMARK. Corollary 2.5 partially answers the following question posed by Tanaka in [2]: Does every separable k'-space with a point-countable k-system have a countable k-system?

THEOREM 2.6. *For a space X, the following are equivalent:*

(1) *X is a compact-covering and quotient cL-image of a paracompact locally compact space.*

(2) *X is a quotient cL-image of a paracompact locally compact space.*

(3) *X has a compact-countable k-system.*

PROOF. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Suppose $f : M \to X$ is a quotient cL-mapping, where M is a paracompact locally compact space. Then M has a locally-finite open cover $\mathcal{B}$ such that for each $B \in \mathcal{B}$, $\overline{B}$ is compact in M. Let $\mathcal{P} = \{ f(\overline{B}) : B \in \mathcal{B} \}$. Since f is a cL-mapping, then $\mathcal{P}$ is a compact-countable cover consisting of compact subsets of X. By virtue of Lemma 1.7 in [6], X is determined by $\mathcal{P}$ because f is a quotient mapping. Thus $\mathcal{P}$ is a compact-countable k-system.

(3) $\Rightarrow$ (1). Let $\mathcal{P}$ be a compact-countable k-system for X. Then X is a k-space. By Proposition 2.1, $\mathcal{P}$ is a k-cover consisting of compact subsets. Put $M = \oplus\mathcal{P}$, and let $f : M \to X$ be the natural map. Then M is a paracompact locally compact space, and f is a cL-mapping. We shall show that f is compact-covering. In fact, for any compact subset K of X, since $\mathcal{P}$ is a k-cover of X, there is a finite $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \cup\mathcal{P}'$. Let $L = \oplus\{K \cap P : P \in \mathcal{P}'\}$. Then L is compact in M with $f(L) = K$, and so f is compact-covering. Because X is a k-space, f is also a quotient mapping. This completes the proof of the theorem.

THEOREM 2.7. *For a space X, we consider the following conditions.*

(1) *X is a sequentially quotient cL-image of a paracompact locally compact space.*

(2) *X has a compact-countable cs^*-cover consisting of compact subsets of X.*

(3) *X is a sequence-covering cL-image of paracompact locally compact space.*

(4) *X is a subsequence-covering cL-image of a paracompact locally compact space.*

Then (1) $\Longleftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If X is also a sequential space, then* (4) $\Rightarrow$ (1).

PROOF. (1) $\Rightarrow$ (2). Assume that M is a paracompact locally compact space, and that $f : M \to X$ is a sequentially quotient cL-mapping. Then M has a locally-finite open cover $\mathcal{B}$ such that for each $B \in \mathcal{B}$, $\overline{B}$ is compact in M. Let $\mathcal{P} = \{ f(\overline{B}) : B \in \mathcal{B} \}$. Since f is a cL-mapping, then $\mathcal{P}$ is a compact-countable cover consisting of compact subsets of X. We shall show that $\mathcal{P}$ is a cs^*-cover of X. In fact, for any sequence $\{x_n\}$ with $x_n \to x \in X$, because f is sequentially-quotient, there are a subsequence $\{ x_{n_i} \}$ of $\{x_n\}$ and some sequence $\{y_i\}$ with $y_i \in f^{-1}(x_{n_i})$ such that $y_i \to y \in f^{-1}(x)$ in M. Thus some $B \in \mathcal{B}$ contains $\{y_i\}$ eventually because $\mathcal{B}$ is an open cover of M. Hence $f(\overline{B}) \in \mathcal{P}$ contains $\{ x_{n_i} \}$ eventually. This shows that $\mathcal{P}$ is a cs^*-cover of X.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{P}$ is a compact-countable cs^*-cover consisting of compact subsets of X. Put $M = \oplus\mathcal{P}$, and let $f : M \to X$ be the natural map. Then M is a paracompact locally compact space, and f is a cL-map. We shall show that f is sequentially quotient. In fact, for any sequence $\{x_n\}$ with $x_n \to x$ in X, denote $S = \{x\} \cup \{x_n : n \in N\}$. Then there is a finite $\mathcal{P}' \subset \mathcal{P}$ such that $S \subset \cup\mathcal{P}'$. Let $L = \oplus\{P \cap S : P \in \mathcal{P}'\}$. Then L is sequen-

tially compact in M with $f(L)=S$, and so there is a convergent sequence L' such that $f(L')$ is a subsequence of S. This shows that f is sequentially quotient.

(2) $\Rightarrow$ (3). From the proof of (2) $\Rightarrow$ (1), we have that L is compact in M with $f(L)=S$.

(3) $\Rightarrow$ (4). Trivial.

Suppose that X is also a sequential space, and $f: M \to X$ is subsequence-covering. From the proof of Lemma 1.6 in [11], f is sequentially quotient. Hence
(4) $\Rightarrow$ (1) holds.

By Proposition 2.2, Theorem 2.6 and Theorem 2.7, we have:

COROLLARY 2.8. *The following are equivalent for a sequential space X:*

(1) *X is a compact-covering and quotient cL-image of a paracompact locally compact space.*

(2) *X is a quotient cL-image of a paracompact locally compact space.*

(3) *X is a sequentially quotient and quotient cL-image of a paracompact locally compact space.*

(4) *X is a sequence-covering and quotient cL-image of a paracompact locally compact space.*

(5) *X is a subsequence-covering and quotient cL-image of a paracompact locally compact space.*

(6) *X has a compact-countable k-system.*

COROLLARY 2.9. *The following are equivalent for a space X:*

(1) *X is a compact-covering and quotient cs-image of a locally compact metric space.*

(2) *X is a quotient cs-image of a locally compact metric space.*

(3) *X is a sequentially quotient and quotient cs-image of a locally compact metric space.*

(4) *X is a sequence-covering and quotient cs-image of a locally compact metric space.*

(5) *X is a subsequence-covering and quotient cs-image of a locally compact metric space.*

(6) *X has a compact-countable mk-system.*

By Corollary 2.9, and Theorem 13 in [12], we have:

COROLLARY 10. *The following are equivalent for a space X:*

(1) *X is a pseudo-open cL-image of a paracompact locally compact space.*

(2) *X is a k'-space with a compact-countable k-system.*

COROLLARY 11. *The following are equivalent for a space X:*

(1) *X is a pseudo-open cs-image of a locally compact metric space.*

(2) *X is a Fréchet space with a compact-countable mk-system.*

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